§5.1 Jointly Distributed Random Variables

Discrete: X & Y defined on same sample space

$$\boxed{\begin{array}{l}\underline{\text{Joint p.m.f.}}\\ p(x,y) = P(X=x \ \& \ Y=y)\end{array}}$$

$p(x,y)$	x_0	x_1	x_2	x_3	Total
y_0					
y_1		$p(x,y)$			$p_Y(y)$
y_2					
Total		$p_X(x)$			1

2-Dimensional Table of Probability Values

- $p(x,y)$ is now a "2-dimensional" table with entries

	$\cdots$	x_k	$\cdots$
$\vdots$			
y_j		$p(x_k,y_j)$	
$\vdots$			

$$p(x_k,y_j) \leftarrow p(x_k,y_j) = P\left(\begin{array}{c}X = x_k \\ \text{and} \\ Y = y_j\end{array}\right)$$

- Recover the usual $P(X=x_k)$ & $P(Y=y_j)$ by adding vertically or horizontally

$$p_X(x_k) = P(X=x_k) = \sum_y p(x_k,y) \quad (\text{sum of column})$$

$$p_Y(y_j) = P(Y=y_j) = \sum_x p(x,y_j) \quad (\text{sum of row})$$

(p_X & p_Y are called "marginal" p.d.f. because they would be written in "Total" row/column at the "margin" of pdf table)

We can also compute conditional p.d.f.

- $p_{X|Y} = \dfrac{p(x,y)}{p_Y(y)} \leftarrow P(X=x \mid Y=y)$
- $p_{Y|X} = \dfrac{p(x,y)}{p_X(x)} \leftarrow P(Y=y \mid X=x)$

Note: The conditional pdf are the rows/columns of $p(x,y)$ renormalized so that they sum to 1.

$p(x,y)$	x_0	x_1	x_2	Total
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_j	$p(x_0,y_j)$	$p(x_1,y_j)$	$p(x_2,y_j)$	$p_Y(y_j)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

X	x_0	x_1	x_2	Total
$p_{X\|Y}(x\|y_j)$	$\dfrac{p(x_0,y_j)}{p_Y(y_j)}$	$\dfrac{p(x_1,y_j)}{p_Y(y_j)}$	$\dfrac{p(x_2,y_j)}{p_Y(y_j)}$	1

Example: Given three "weighted" coins, experiment is to pick random coin, flip it three times & count # H.

Suppose $P(H \mid \text{Coin } 1) = 2/10$

$P(H \mid \text{Coin } 2) = 5/10$

$P(H \mid \text{Coin } 3) = 7/10$

Prob	Coin 1	Coin 2	Coin 3	Total
0				
1		$p(x,y)$		
2				$p_Y(y)$
3				
Total		$p_X(x)$		1

(# Heads)

X = # of chosen coin $\begin{cases} \text{Coin } 1 \to 1 \\ \text{Coin } 2 \to 2 \\ \text{Coin } 3 \to 3 \end{cases}$

Y = # of H counted

Interior of table: $p(x,y) = P\left(\begin{matrix}\text{choose coin } x \\ \text{and} \\ \text{flip } y \text{ H with coin}\end{matrix}\right)$

Margins of table: $p_X(x) = P(\text{choose coin } x)$

$p_Y(y) = P(\text{flip } y \text{ H})$

Rows and Columns (renormalized) are conditionals

Ex: The $y=2$ row gives the conditional

$$p_{X|Y}(x|2) = P\left(\text{chose coin } x \mid \text{flipped } 2 \text{ H}\right)$$

Ex: The $x=1$ column gives the conditional

$$p_{Y|X}(y|1) = P\left(\text{flip } y \text{ H} \mid \text{chose coin } 1\right)$$

As before we can discuss "Independence"

> **Def:** The random variables X & Y are *independent* if
>
> $$p(x,y) = p_X(x) \cdot p_Y(y)$$

Note: This is equivalent to

- $p_{X|Y}(x|y) = p_X(x)$
- $p_{Y|X}(y|x) = p_Y(y)$

Numerical Example. Given joint p.m.f.

$p(x,y)$		$X=-1$	$X=0$	$X=1$
Y	2	2/15	2/15	1/15
	1	2/15	3/15	2/15
	0	1/15	1/15	1/15

Compute marginal p.m.f. and conditional p.m.f.

Note: Values in pmf table record probabilities:

$$P(X=0 \;\&\; Y=1) = p(0,1) = 3/15$$

Marginal pmf are given by computing "Totals"

↳ Summing either rows or columns

$p(x,y)$		$X=-1$	$X=0$	$X=1$	Total
Y	2	2/15	2/15	1/15	5/15
	1	2/15	3/15	2/15	7/15
	0	1/15	1/15	1/15	3/15
Total		5/15	6/15	4/15	

X-marginal = "vertical" sums

X	-1	0	1
$p_X(x)$	5/15	6/15	4/15

Y-marginal = "horizontal" sums

Y	0	1	2
$p_Y(y)$	3/15	7/15	5/15

Conditional pmf are given by dividing by marginal

↳ "Renormalize" row or column to sum to 1.

$p(x,y)$		$X=-1$	$X=0$	$X=1$	Total
Y	2	2/15	2/15	1/15	
	1	2/15	3/15	2/15	7/15
	0	1/15	1/15	1/15	
Total			6/15		

$Y=1$ $X=0$

$(X|Y)$ = rows

X	-1	0	1
$p_{X\mid Y}(x\mid 1)$	2/7	3/7	2/7

$$3/7 = \frac{3/15}{7/15} = \frac{p(0,1)}{p_Y(1)}$$

$(Y|X)$ = columns

Y	0	1	2
$p_{Y\mid X}(y\mid 0)$	1/6	3/6	2/6

$$1/6 = \frac{1/15}{6/15} = \frac{p(0,0)}{p_X(0)}$$

Note: In this problem, X & Y are not independent because for example

$$p(0,1) \;\;\; p_X(0)\cdot p_Y(1)$$
$$\| \qquad\qquad \|$$
$$3/15 \neq 6/15 \cdot 7/15$$

§5.1 Jointly Distributed Random Variables

Continuous: X & Y both defined on $\mathbb{R}$

Joint p.d.f.

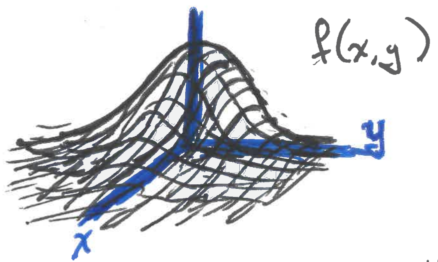
$f(x, y)$ ← function of two variables!

Probability is given by integrating
(Double Integral over a Region)

$$P\big((X, Y) \text{ in } R\big) = \iint_R f(x,y)\, dA$$

R is some region in 2-dimensions

R 



Marginal & Conditional pdf are defined the same as before... with $\int$ instead of Σ

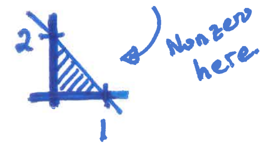
Marginal pdf: Integrate "horizontally" or "vertically"

- $f_X(x) = \int_{y \in \mathbb{R}} f(x,y)\, dy$ ← "sum of column"
- $f_Y(y) = \int_{x \in \mathbb{R}} f(x,y)\, dx$ ← "sum of row"

Conditional pdf: Renormalize "horizontally" or "vertically"

- $f_{X|Y}(x \mid y) = \dfrac{f(x,y)}{f_Y(y)}$
- $f_{Y|X}(y \mid x) = \dfrac{f(x,y)}{f_X(x)}$

Example: Suppose $f(x,y) = 6xy$ for $\begin{cases} x, y \geq 0 \\ 2x + y \leq 2 \end{cases}$



I) Compute marginal p.d.f.

$$f_X(x) = \int_y 6xy\, dy$$

$$= \int_{y=0}^{y=2-2x} 6xy\, dy = 3xy^2 \Big|_{y=0}^{y=2-2x}$$

$$= 3x(2-2x)^2 = 12x - 24x^2 + 12x^3$$

Note: $\int_x f_X(x)\, dx = \int_0^1 12x - 24x^2 + 12x^3\, dx = 12/2 - 24/3 + 12/4$
$= 1$

$$f_Y(y) = \int_x 6xy\, dx$$

$$= \int_{x=0}^{x=1-\frac{1}{2}y} 6xy\, dx = \cdots = 3y\left(1 - \tfrac{1}{2}y\right)^2$$

II) Compute conditional p.d.f.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{6xy}{3y(1-\frac{1}{2}y)^2}$$

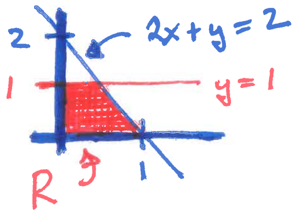
$$= 2\, x/(1-\tfrac{1}{2}y)^2$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{6xy}{3x(2-2x)^2}$$

$$= 2\, y/(2-2x)^2$$

III) Compute $P(Y \le 1)$

[This can be computed either as a double integral of $f(x,y)$ or a single integral of $f_Y(y)$.]



$$\iint_R f(x,y)\, dA = \int_{y \le 1} f_Y(y)\, dy$$

$$P(Y \le 1) = \iint_R 6xy \, dA$$

do dx integral first....

$$= \int_{y=0}^{y=1} \underbrace{\int_{x=0}^{x=1-\frac{1}{2}y} 6xy \, dx}_{} \, dy$$

Note: This part is $f_Y(y)$!!

$$= \int_{y=0}^{y=1} 3y(1-\tfrac{1}{2}y)^2 \, dy$$

$$= \int_{y=0}^{y=1} 3y - 3y^2 + \tfrac{3}{4}y^3 \, dy$$

$$= \tfrac{3}{2}y^2 - \tfrac{3}{3}y^3 + \tfrac{3}{16}y^4 \Big|_0^1$$

$$= \tfrac{3}{2} - 1 + \tfrac{3}{16} = \tfrac{7}{16}$$